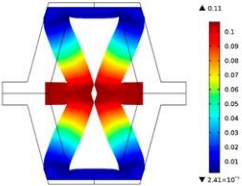

Multiphysics
Mechanics of Solid

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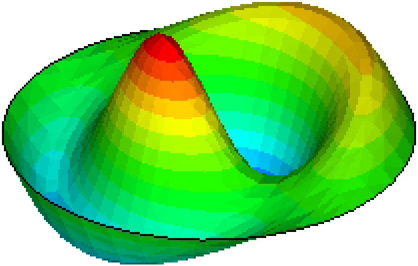
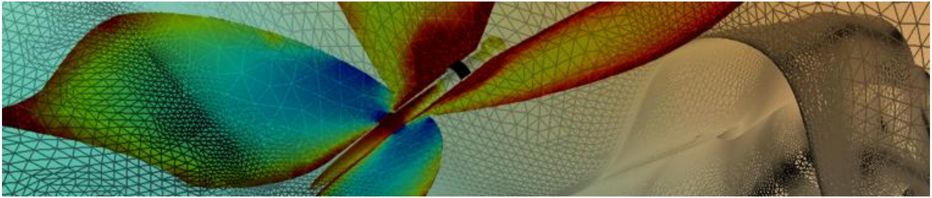
DATE: 2020.05.12



Allan F. Bower

Applied Mechanics of Solids

http://solidmechanics.org/index.html



Introduction to Structural Mechanics

https://www.comsol.com/multiphysics/introduction-to-structural-mechanics

WHAT IS MULTIPHYSICS?

Multiphysics *mul·ti·phys·ics [mul-ti-fiz-iks] noun*

1. Coupled physical phenomena in computer simulation.
2. The study of multiple interacting physical properties.

Understanding Physics

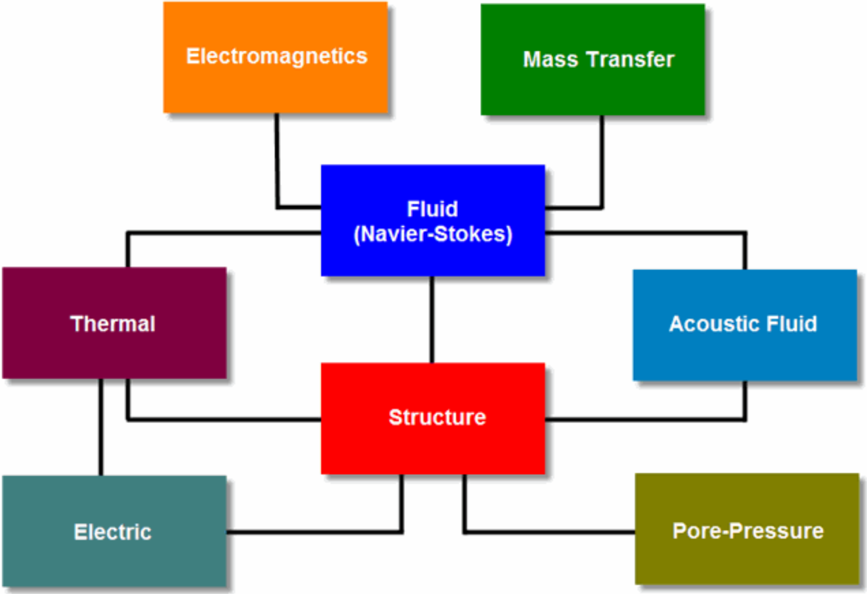
We can describe what happens in the world using sets of physical laws. Since the 1940s, we have been using computers to understand physical phenomena. Originally, computing resources were scarce, so physical effects were observed in isolation. But, as we know, physics do not occur in isolation in the real world.

IT'S A MULTIPHYSICS WORLD

The real world is multiphysics in nature.

Consider your cell phone. The antenna receives electromagnetic waves, the touch screen or buttons are mechanical and electrical components that interact with each other, the battery involves chemical reactions and the movement of ions and electrical current, and so on. A single device, but multiphysics.

With a multiphysics capable simulation tool, you can correctly capture the important aspects of your design.


Electromagnetics
Mass Transfer
Fluid
(Navier-Stokes)
Thermal
Acoustic Fluid
Structure
Electric
Pore-Pressure

MATERIALS MODELS

The material models can be grouped by families, are as follows:

A) **Linear Elastic Material**;
B) **Nonlinear Elastic Material Models**: Ramberg-Osgood, Power Law, Bilinear Elastic, Uniaxial Data, Hyperbolic Law, Hardin-Drnevich, Duncan-Chang, Duncan-Selig;
C) **Linear Viscoelastic Materials**: Generalized Maxwell Model, Standard Linear Solid Model, Kelvin-Voigt Model;
D) **Hyperelastic Material Models**: Neo-Hookean, St Venant-Kirchhoff, Mooney-Rivlin, Two Parameters, Mooney-Rivlin, Five Parameters, Mooney-Rivlin, Nine Parameters, Yeoh, Ogden, Storakers, Varga, Arruda-Boyce, Arruda-Boyce, Blatz-Ko, Gao, Murnaghan;

E) **Elastoplastic Material Models**: von Mises Criterion, Tresca Criterion, Mohr-Coulomb Criterion, Drucker-Prager Criterion, Matsuoka-Nakai Criterion, Lade-Duncan Criterion, Hill Orthotropic Plasticity;

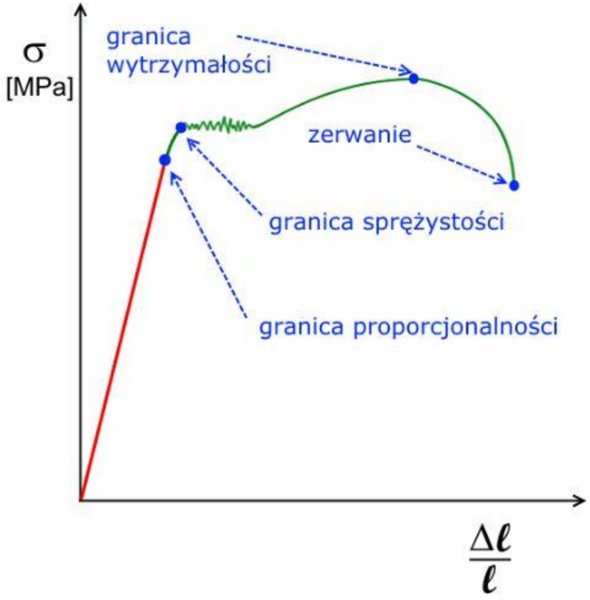
F) **Failure Criteria** for Concrete, Rocks, and Other Brittle Material: Bresler-Pister Criterion, Willam-Warnke Criterion, Ottosen Criterion, Original Hoek-Brown Criterion, Generalized Hoek-Brown Criterion;

G) **Cam-Clay** Material Model;

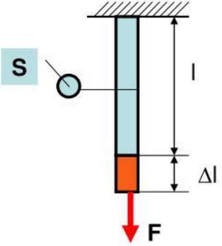
H) **Creep and Viscoplasticity**: Norton Law (Power law), Norton-Bailey Law, Garofalo Law (Hyperbolic Sine Law), Navarro-Herring Creep (Diffusional Creep), Coble Creep (Diffusional Creep), Weertman Creep (Dislocation Creep), Anand Viscoplastic Model;

I) **Piezoelectric Material**;

J) **Rigid Domain** Material Model.


σ
[MPa]
granica
wytrzymałości
zerwanie
granica sprężystości
granica proporcjonalności
Δℓ
ℓ

Prawo Hooke'a



$$\Delta l = \frac{Fl}{ES}$$

gdzie: Δl – wydłużenie
F – siła,
l – długość początkowa,
E – moduł Younga,
S – pole przekroju poprzecznego

Współczynnik proporcjonalności E – moduł sprężystości przy rozciąganiu (Moduł Younga, 1807)

$$\varepsilon = \frac{\Delta l}{l} \quad \sigma = \frac{F}{S} \quad \sigma = \varepsilon E$$

TENSOR NOTATION

Some of the theory is developed using tensor notation. In most cases, explicit index notation is avoided. This means that the order of a tensor usually must be understood from the context. As an example, Hooke's law for linear elasticity is usually written like

$$\mathbf{S} = \mathbf{C} : \boldsymbol{\varepsilon}.$$

Here, the stress tensor $\mathbf{S}$ and the strain tensor $\boldsymbol{\varepsilon}$ are second order tensors, while the constitutive tensor $\mathbf{C}$ is a fourth order tensor. The ':' symbol means a contraction over two indices.

In a notation where the indices are shown, the same equation would read

$$S_{ij} = C_{ijkl} : \varepsilon_{kl},$$

where the Einstein summation convention has been used as a shorthand for

$$S_{ij} = \sum_{k=1}^{3} \sum_{l=1}^{3} C_{ijkl} \varepsilon_{kl}.$$

In a few cases, non-orthonormal coordinate systems must be considered. It is then necessary to keep track of the covariance and contravariance properties of tensors. In such a case, Hooke's law is written

$$S^{ij} = C^{ijkl} : \varepsilon_{kl}$$

The stress and constitutive tensors have contravaraint components, while the strain tensor has covariant components.

The elastic strain energy density is

$$W_{\mathrm{S}} = \frac{1}{2}\boldsymbol{\varepsilon}_{\mathrm{el}}:(\mathbf{C}:\boldsymbol{\varepsilon}_{\mathrm{el}} + \mathbf{2S}_0) = \frac{1}{2}\boldsymbol{\varepsilon}_{\mathrm{el}}:(\mathbf{S} + \mathbf{2S}_0).$$

This expression assumes that the initial stress contribution is constant during the straining of the material.

Due to the symmetry, the elasticity tensor can be completely represented by a symmetric 6-by-6 matrix as:

$$\mathbf{C} = \begin{bmatrix} D_{11} & D_{12} & D_{13} & D_{14} & D_{15} & D_{16} \\ D_{21} & D_{22} & D_{23} & D_{24} & D_{25} & D_{26} \\ D_{31} & D_{32} & D_{33} & D_{34} & D_{35} & D_{36} \\ D_{41} & D_{42} & D_{43} & D_{44} & D_{45} & D_{46} \\ D_{51} & D_{52} & D_{53} & D_{54} & D_{55} & D_{56} \\ D_{61} & D_{62} & D_{63} & D_{64} & D_{65} & D_{66} \end{bmatrix}$$

$$= \begin{bmatrix} \mathrm{C}^{1111} & \mathrm{C}^{1122} & \mathrm{C}^{1133} & \mathrm{C}^{1112} & \mathrm{C}^{1123} & \mathrm{C}^{1113} \\ \mathrm{C}^{1122} & \mathrm{C}^{2222} & \mathrm{C}^{2233} & \mathrm{C}^{2212} & \mathrm{C}^{2223} & \mathrm{C}^{2213} \\ \mathrm{C}^{1133} & \mathrm{C}^{2233} & \mathrm{C}^{3333} & \mathrm{C}^{3312} & \mathrm{C}^{3323} & \mathrm{C}^{3313} \\ \mathrm{C}^{1112} & \mathrm{C}^{2212} & \mathrm{C}^{3312} & \mathrm{C}^{1212} & \mathrm{C}^{1223} & \mathrm{C}^{1213} \\ \mathrm{C}^{1123} & \mathrm{C}^{2223} & \mathrm{C}^{3323} & \mathrm{C}^{1223} & \mathrm{C}^{2323} & \mathrm{C}^{2313} \\ \mathrm{C}^{1113} & \mathrm{C}^{2213} & \mathrm{C}^{3313} & \mathrm{C}^{1213} & \mathrm{C}^{2313} & \mathrm{C}^{1313} \end{bmatrix}.$$

In the case of the isotropic material and elastic moduli the elasticity matrix becomes

$$\mathbf{C} = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}.$$

Different pairs of elastic moduli can be used, and as long as two moduli are defined, the others can be computed according to Table 1.

According to Table 1, the elasticity matrix for isotropic materials can be written in terms of Lamé parameters λ and μ, as follows:

$$\mathbf{C} = \begin{bmatrix} \lambda + 2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda + 2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda + 2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix}.$$

Stress-strain relations for linear elastic <u>orthotropic</u> materials

An orthotropic material has three mutually perpendicular symmetry planes. This type of material has 9 independent material constants. With basis vectors perpendicular to the symmetry plane, the elastic stiffness matrix has the form:

$$\mathbf{C} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{22} & c_{23} & 0 & 0 & 0 \\ c_{13} & c_{23} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{bmatrix}$$

A.F. Bower. *Applied Mechanics of Solids.* CRC Press, Boca Raton, FL, **2010**.
http://solidmechanics.org/Text/Chapter3_2/Chapter3_2.php

The engineering constants are related to the components of the compliance tensor by

$$c_{11} = E_1(1 - \nu_{23}\nu_{32})\Gamma, c_{22} = E_2(1 - \nu_{13}\nu_{31})\Gamma, c_{33} = E_3(1 - \nu_{12}\nu_{21})\Gamma,$$

$$c_{12} = E_1(\nu_{21} + \nu_{31}\nu_{23})\Gamma = E_2(\nu_{12} + \nu_{32}\nu_{13})\Gamma,$$

$$c_{13} = E_1(\nu_{31} + \nu_{21}\nu_{32})\Gamma = E_3(\nu_{13} + \nu_{12}\nu_{23})\Gamma,$$

$$c_{23} = E_2(\nu_{32} + \nu_{12}\nu_{31})\Gamma = E_3(\nu_{23} + \nu_{21}\nu_{13})\Gamma,$$

$$c_{44} = \mu_{23},\ c_{55} = \mu_{13}, c_{66} = \mu_{12},$$

$$\Gamma = \frac{1}{1 - \nu_{12}\nu_{21} - \nu_{23}\nu_{32} - \nu_{13}\nu_{31} - 2\nu_{21}\nu_{32}\nu_{13}}.$$

Stress-strain relations for linear elastic Transversely Isotropic Material

A special case of an orthotropic solid is one that contains a plane of isotropy (this implies that the solid can be rotated with respect to the loading direction about one axis without measurable effect on the solid's response). Choose e3 perpendicular to this symmetry plane. Then, transverse isotropy requires that: $c_{22}=c_{11}$, $c_{23}=c_{13}$, $c_{55}=c_{44}$, $c_{66}=(c_{11}-c_{12})/2$, so that the stiffness matrix has the form

$$\mathbf{C} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{11} & c_{13} & 0 & 0 & 0 \\ c_{13} & c_{13} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & (c_{11} - c_{12})/2 \end{bmatrix}$$

The engineering constants must satisfy

$$E_1 = E_2 = E_p \quad E_3 = E_t$$

$$\nu_{12} = \nu_{21} = \nu_p \quad \nu_{31} = \nu_{32} = \nu_{tp}$$

$$\nu_{13} = \nu_{23} = \nu_{pt}$$

The engineering constants and stiffnesses are related by

$$c_{11} = c_{22} = E_p\left(1 - \nu_{pt}\nu_{tp}\right)\Upsilon \quad c_{33} = E_t\left(1 - \nu_p^2\right)\Upsilon \quad c_{12} = E_p\left(\nu_p + \nu_{pt}\nu_{tp}\right)\Upsilon$$

$$c_{13} = c_{23} = E_p\left(\nu_{tp} + \nu_p\nu_{tp}\right)\Upsilon = E_t\left(\nu_{pt} + \nu_p\nu_{pt}\right)\Upsilon \qquad c_{44} = \mu_t \qquad c_{66} = \mu_p$$

$$\Upsilon = \frac{1}{1-\nu_p^2-2\nu_{pt}\nu_{tp}-2\nu_p\nu_{pt}\nu_{tp}}$$

$$E_p = \left(c_{11}^2 c_{33} + 2c_{13}^2 c_{12} - 2c_{11}c_{13}^2 - c_{33}c_{12}^2\right) / \left(c_{11}c_{33} - c_{13}^2\right)$$

$$E_t = \left(c_{11}^2 c_{33} + 2c_{13}^2 c_{12} - 2c_{11}c_{13}^2 - c_{33}c_{12}^2\right) / \left(c_{11}^2 - c_{12}^2\right)$$

$$\nu_p = (c_{12}c_{33} - c_{13}^2)/(c_{11}c_{33} - c_{13}^2), \qquad \nu_{tp} = (c_{13}c_{11} - c_{12}c_{13})/(c_{11}^2 - c_{12}^2)$$

$$\nu_{pt} = (c_{11}c_{13} - c_{12}c_{13})/(c_{11}c_{33} - c_{13}^2), \quad \mu_{23} = c_{44}, \quad \mu_{13} = c_{55} \quad \mu_{12} = c_{66}$$

Table 1. Expressions for elastic moduli.

DESCRIPTION	VARIABLE	$D(E,\nu)$	$D(K,G)$	$D(\lambda,\mu)$
Young's modulus	E		$\frac{9KG}{3K+G}$	$\mu\frac{3\lambda+2\mu}{\lambda+\mu}$
Poisson's ratio	ν		$\frac{1}{2}\left(1-\frac{3G}{3K+G}\right)$	$\frac{\lambda}{2(\lambda+\mu)}$
Bulk modulus	K	$\frac{E}{3(1-2\nu)}$		$\lambda+\frac{2\mu}{3}$
Shear modulus	G	$\frac{E}{2(1+\nu)}$		μ
Lamé parameter λ	λ	$\frac{E\nu}{(1+\nu)(1-2\nu)}$	$K-\frac{2G}{3}$	
Lamé parameter μ	μ	$\frac{E}{2(1+\nu)}$	G	
Pressure-wave speed	c_p		$\sqrt{\frac{K+4G/3}{\rho}}$	
Shear-wave speed	c_s		$\sqrt{G/\rho}$	

Physical Interpretation of elastic constants for isotropic solids

It is important to have a feel for the physical significance of the two elastic constants E and v.

Young's modulus E is the slope of the stress— strain curve in uniaxial tension. It has dimensions of stress ($N/m2$) and is usually large – for steel, $E=210\times10^9\ N/m^2$. You can think of E as a measure of the stiffness of the solid. The larger the value of E, the stiffer the solid. For a stable material, $E>0$.

Poisson's ratio v is the ratio of lateral to longitudinal strain in uniaxial tensile stress. It is dimensionless and typically ranges from 0.2—0.49, and is around 0.3 for most metals. For a stable material, $-1<v<0.5$. It is a measure of the compressibility of the solid. If $v=0.5$, the solid is incompressible – its volume remains constant, no matter how it is deformed. If $v=0$, then stretching a specimen causes no lateral contraction. Some bizarre materials have $v<0$ - if you stretch a round bar of such a material, the bar increases in diameter!

Thermal expansion coefficient ***quantifies the change in volume of a material if it is heated in the absence of stress. It has dimensions of (degrees Kelvin)$^{-1}$ and is usually very small.***
For steel, $\alpha \approx 6-10 \times 10-6\, K\text{-}1$

The **bulk modulus** quantifies the resistance of the solid to volume changes. It has a large value (usually bigger than *E*).

The **shear modulus** quantifies its resistance to volume preserving shear deformations. Its value is usually somewhat smaller than *E*.

	LAME MODULUS λ	SHEAR MODULUS μ	YOUNG'S MODULUS E	POISSON'S RATIO ν	BULK MODULUS K
λ, μ			$\frac{\mu(3\lambda+2\mu)}{\lambda+\mu}$	$\frac{\lambda}{2(\lambda+\mu)}$	$\frac{3\lambda+2\mu}{3}$
λ, E		Irrational		Irrational	Irrational
λ, ν		$\frac{\lambda(1-2\nu)}{2\nu}$	$\frac{\lambda(1+\nu)(1-2\nu)}{\nu}$		$\frac{\lambda(1+\nu)}{3\nu}$
λ, K		$\frac{3(K-\lambda)}{2}$	$\frac{9K(K-\lambda)}{3K-\lambda}$	$\frac{\lambda}{3K-\lambda}$	
μ, E	$\frac{\mu(2\mu-E)}{E-3\mu}$			$\frac{E-2\mu}{2\mu}$	$\frac{\mu E}{3(3\mu-E)}$
μ, ν	$\frac{2\mu\nu}{1-2\nu}$		$2\mu(1+\nu)$		$\frac{2\mu(1+\nu)}{3(1-2\nu)}$
μ, K	$\frac{3K-2\mu}{3}$		$\frac{9K\mu}{3K+\mu}$	$\frac{3K-2\mu}{2(3K+\mu)}$	
E, ν	$\frac{\nu E}{(1+\nu)(1-2\nu)}$	$\frac{E}{2(1+\nu)}$			$\frac{E}{3(1-2\nu)}$
E, K	$\frac{3K(3K-E)}{9K-E}$	$\frac{3EK}{9K-E}$		$\frac{3K-E}{6K}$	
ν, K	$\frac{3K\nu}{(1+\nu)}$	$\frac{3K(1-2\nu)}{2(1+\nu)}$	$3K(1-2\nu)$		

POISSON'S RATIO

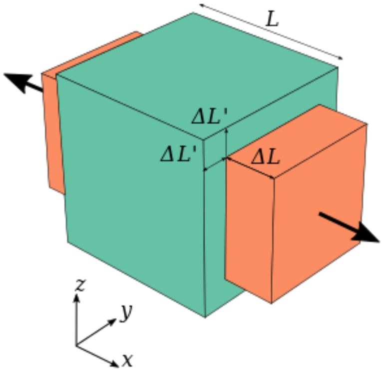


Figure 1. A cube with sides of length *L* of an isotropic linearly elastic material subject to tension along the x axis, with a Poisson's ratio of 0.5.

The green cube is unstrained, the red is expanded in the *x* direction by ΔL due to tension, and contracted in the *y* and *z* directions by $\Delta L'$.

For a cube stretched in the *x*-direction (see Figure 1) with a length increase of ΔL in the *x* direction, and a length decrease of $\Delta L'$ in the *y* and *z* directions, the infinitesimal diagonal strains are given by

$$d\varepsilon_x = \frac{dx}{x}, d\varepsilon_x = \frac{dy}{y}, d\varepsilon_z = \frac{dz}{z}.$$

If Poisson's ratio is constant through deformation, integrating these expressions and using the definition of Poisson's ratio gives

$$-\nu \int_L^{L+\Delta L} \frac{dx}{x} = \int_L^{L+\Delta L'} \frac{dy}{y} = \int_L^{L+\Delta L'} \frac{dz}{z} .$$

Solving and exponentiating, the relationship between ΔL and $\Delta L'$ is then

$$\left(1+\frac{\Delta L}{L}\right)^{-\nu}=1+\frac{\Delta L'}{L}.$$

For very small values of ΔL and $\Delta L'$, the first-order approximation yields:

$$\nu \approx -\frac{\Delta L'}{\Delta L}.$$

Without approximation we can define Poisson's ratio as:

$$\nu = -\frac{log\left(1+\frac{\Delta L'}{L}\right)}{log\left(1+\frac{\Delta L}{L}\right)}.$$

Volumetric change

The relative change of volume $\Delta V/V$ of a cube due to the stretch of the material can now be calculated. Using $V = L^3$ and $V + \Delta V = (L + \Delta L)(L + \Delta L')^2$:

$$\frac{\Delta V}{V} = \left(1 + \frac{\Delta L}{L}\right)\left(1 + \frac{\Delta L'}{L}\right)^2 - 1$$

Using the above derived relationship between ΔL and $\Delta L'$:

$$\frac{\Delta V}{V} = \left(1 + \frac{\Delta L}{L}\right)^{1-2\nu} - 1$$

for very small values of ΔL and $\Delta L'$, the first-order approximation yields:

$$\frac{\Delta V}{V} = (1 - 2\nu)\frac{\Delta L}{L}.$$

For isotropic materials we can use Lamé’s relation

$$\nu \approx \frac{1}{2} - \frac{E}{6K}$$

where K is bulk modulus and is E elastic modulus (or Young's modulus).

Note that isotropic materials must have a Poisson's ratio of $-1 < \nu < 0.5$. Typical isotropic engineering materials have a Poisson's ratio of $0.2 < \nu < 0.5$.

Width change

If a rod with diameter (or width, or thickness) d and length L is subject to tension so that its length will change by ΔL then its diameter d will change by:

$$\Delta d = -d\,\nu\,\frac{\Delta L}{L}.$$

The above formula is true only in the case of small deformations; if deformations are large then the following (more precise) formula can be used:

$$\Delta d = -d\left(1-\left(1+\frac{\Delta L}{L}\right)^{-\nu}\right).$$

The value is negative because it decreases with increase of length

Definition of PR

The most common definition of the engineering Poisson's ratio (PR) is based on the assumption of small deformation. PR is simply defined as a negative ratio of the transverse to longitudinal strains. More generally the Poisson's ratio for the longitudinal direction **l** and the transverse direction **t** can be written [Woj2005]

$$\nu_{lt} = -\frac{\varepsilon_{tt}}{\varepsilon_{ll}}, \qquad ()$$

where ε_{tt} and ε_{ll} are strains in transverse and longitudinal direction, respectively.

In the case of non-homogeneous material, the homogenization technique is used. The **effective value of the Poisson's ratio** is defined as a negative ratio of the average transverse to longitudinal strains:

$$\nu_{eff} = -\frac{\langle\varepsilon_t\rangle}{\langle\varepsilon_l\rangle}\,, \qquad ()$$

where $\langle\varepsilon_t\rangle$ and $\langle\varepsilon_l\rangle$ are average strains in transverse and longitudinal direction, respectively.

In the case of a large deformation, however, the expression describing effective PR might require more complex, nonlinear form. The logarithmic PR model is expressed by the following formulae:

$$\nu_{eff} = -\frac{\log(1+\langle\varepsilon_t\rangle)}{\log(1+\langle\varepsilon_l\rangle)}, \qquad ()$$

but other models could also be considered.

linear isotropic elasticity.

Material	**Poisson's ratio**
Isotropic upper limit [1]	0.5
Rubber [6]	0.48- ~0.5
Indium [11]	0.45
Gold [4]	0.42
Lead [4]	0.44
Copper [7]	0.37
Aluminum [4]	0.34
Copper [4]	0.35
Polystyrene [6]	0.34
Brass [1]	0.33
Ice [8]	0.33
Polystyrene foam [6]	0.3
Stainless Steel [7]	0.30
Steel [1]	0.29
Tungsten [4]	0.30
Tungsten	0.28
Fused quartz [9]	0.17
Boron [12]	0.08
Beryllium [4]	0.03
Re-entrant foam [10]	-0.7
Isotropic lower limit [1]	-1

References

[1] I. S. Sokolnikoff, Mathematical theory of elasticity. Krieger, Malabar FL, second edition, 1983.

[2] A .M. James and M. P. Lord in Macmillan's Chemical and Physical Data, Macmillan, London, UK, 1992.

[3] G.W.C. Kaye and T.H. Laby in Tables of physical and chemical constants, Longman, London, UK, 15th edition, 1993.

[4] G.V. Samsonov (Ed.) in Handbook of the physicochemical properties of the elements, IFI-Plenum, New York, USA, 1968.

[5] G. Simmons, and H. Wang, Single crystal elastic constants and calculated aggregate properties: a handbook, MIT Press, Cambridge, 2nd ed, 1971.

[6] J. A. Rinde, Poisson's ratio for rigid plastic foams, J. Applied Polymer Science, 14, 1913-1926, 1970.

[7] D. E. Gray, American Institute of Physics Handbook, 3rd ed., chapter 3, McGraw hill, New York, 1973.

[8] E. M. Schulson, The Structure and Mechanical Behavior of Ice, JOM, 51 (2) pp. 21-27, 1999. article link

[9] H. H. Demarest, Jr., Cube resonance method to determine the elastic constants of solids, J. Acoust. Soc. Am. 49, 768-775 (1971).

[10] R. S. Lakes, Foam structures with a Negative Poisson's ratio, Science, 235 1038-1040, 1987.

[11] D. Li, T. M. Jaglinski, D. S. Stone, and R. S. Lakes, Temperature insensitive negative Poisson's ratios in isotropic alloys near a morphotropic phase boundary, Appl. Phys. Lett, 101, 251903, Dec. (2012).

[12] K. A. Gschneidner, Jr., Physical Properties and Interrelationships of Metallic and Semimetallic Elements, Solid State Physics, 16, 275-426, 1964

Mathematical model

The Navier's equation of motion with the linear constitutive relation between stresses and deformations is:

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} - \nabla \cdot \mathbf{S} = \mathbf{F}_\mathrm{V} \qquad (1)$$

where: ρ is the density, $\mathbf{u}$ is the vector of displacements, $\mathbf{S}$ is the stress tensor, $\mathbf{F}_\mathrm{V}$ is the volume force vector.

The total stress $\boldsymbol{S}$ in the Hooke's law is then augmented by the viscoelastic stress $\boldsymbol{S}_{\mathrm{q}}$ and the external stress $\boldsymbol{S}_{\mathrm{ext}}$

$$\mathbf{S} = \boldsymbol{S}_{\mathrm{ad}} + \mathbf{C}:\boldsymbol{\varepsilon}_{\mathrm{el}}$$

$$\boldsymbol{S}_{\mathrm{ad}} = \boldsymbol{S}_{0} + \boldsymbol{S}_{\mathrm{ext}} + \boldsymbol{S}_{\mathrm{q}}$$

of the material models will compute a stress Many based on an elastic strain. The elastic strain tensor is obtained after removing any inelastic deformation contribution from the total deformation from the displacements. There are several possible inelastic strain contributions: initial, thermal, hygroscopic, plastic, creep and viscoplastic strains.

The elastic strain tensor $\boldsymbol{\varepsilon}_{\mathrm{el}}$ represents the total strain minus initial and inelastic strains

$$\boldsymbol{\varepsilon}_{\mathrm{el}} = \boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_{\mathrm{inel}}$$

$$\boldsymbol{\varepsilon} = \frac{1}{2}\left((\nabla\mathbf{u})^T + \nabla\mathbf{u}\right).$$

Initial and inelastic strains $\boldsymbol{\varepsilon}_{\mathrm{inel}} = \boldsymbol{\varepsilon}_0 + \boldsymbol{\varepsilon}_{\mathrm{th}} + \boldsymbol{\varepsilon}_{\mathrm{hs}} + \boldsymbol{\varepsilon}_{\mathrm{pl}} + \boldsymbol{\varepsilon}_{\mathrm{cr}} + \boldsymbol{\varepsilon}_{\mathrm{vp}}$, where strains $\boldsymbol{\varepsilon}_0, \boldsymbol{\varepsilon}_{\mathrm{th}}, \boldsymbol{\varepsilon}_{\mathrm{hs}}, \boldsymbol{\varepsilon}_{\mathrm{pl}}, \boldsymbol{\varepsilon}_{\mathrm{cr}}$ are initial, thermal, hygroscopic, plastic, creep and viscoplastic strains, respectively. This additive decomposition of strains can however only be justified as long as the strains are small. In the case of large deformations, the different strain contributions may not even be commutative.

The elastic strain tensor can in the same way be decomposed into volumetric and deviatoric components: $\boldsymbol{\varepsilon}_{\mathrm{el}} = \frac{1}{3}\boldsymbol{\varepsilon}_{\mathrm{vol}}\mathbf{I} + \boldsymbol{\varepsilon}_{\mathrm{dev}}$, with the volumetric elastic strain given by $\boldsymbol{\varepsilon}_{\mathrm{vol}} = \mathrm{trace}(\boldsymbol{\varepsilon}_{\mathrm{el}})$ and the deviatoric contribution by $\boldsymbol{\varepsilon}_{\mathrm{dev}} = \mathrm{dev}(\boldsymbol{\varepsilon}_{\mathrm{el}})$.

A harmonic displacement is defined by equation as below:

$$\frac{\partial^2 \mathbf{u}}{\partial t^2} = -\omega^2 \mathbf{u} \tag{}$$

where: ω is forcing frequency.

The displacement vector has the complex form and is defined as:

$$\mathbf{u}(\mathbf{x}) = \mathbf{u}_1(\mathbf{x}) + i\mathbf{u}_2(\mathbf{x}) \tag{x}$$

and the harmonic displacement is a real part of complex form:

$$\mathbf{u}(\mathbf{x}, t) = \mathrm{Re}\left[\mathbf{u}(\mathbf{x})\mathrm{e}^{-i\omega t}\right] \tag{x}$$

According to aforementioned equations the harmonic equation of motion fulfills the formula:

$$\rho\omega^2\mathbf{u} - \nabla\cdot\mathbf{S} = \mathbf{F}\mathrm{e}^{i\phi}.$$

$$\mathbf{S} = \mathbf{S}_{\mathrm{ad}} + \mathbf{C}:\boldsymbol{\varepsilon}_{\mathrm{el}} - \left(\frac{\mathrm{trace}(\mathbf{C}:\boldsymbol{\varepsilon}_{\mathrm{el}})}{3} + p_w\right)\mathbf{I}$$

The trace of an n-by-n square matrix **A** is defined as the sum of the elements on the main diagonal (the diagonal from the upper left to the lower right): $\mathrm{trace}(\mathbf{A}) = \sum_i \mathrm{A}_{\mathrm{ii}}$.

The Navier's equation of motion with the linear constitutive relation between stresses and deformations is:

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} - (\mu \nabla^2 \mathbf{u} + (\lambda + \mu) \nabla \nabla \cdot \mathbf{u}) = \mathbf{0}. \qquad ()$$

A harmonic displacement is defined by an equation as below:

$$\frac{\partial^2 \mathbf{u}}{\partial t^2} = -\omega^2 \mathbf{u} \qquad ()$$

where: ω is forcing frequency. The displacement vector has the complex form and is defined as:

$$\mathbf{u}(\mathbf{x}) = \mathbf{u}_1(\mathbf{x}) + i\mathbf{u}_2(\mathbf{x}) \qquad ()$$

and the harmonic displacement is a real part of the complex form:

$$\mathbf{u}(\mathbf{x}, t) = Re\left[\mathbf{u}(\mathbf{x})\mathrm{e}^{-i\omega t}\right] \tag{9}$$

According to aforementioned equations the harmonic equation of motion of linear elastic material fulfills a formula:

$$-\rho\omega^2\mathbf{u} - (\mu\nabla^{\mathbf{2}}\mathbf{u} + (\lambda + \mu)\nabla\nabla \cdot \mathbf{u}) = \mathbf{0} \tag{10}$$

where: μ, λ are Lamé constants. The harmonic equation may be viewed as the eigenvalue equation.

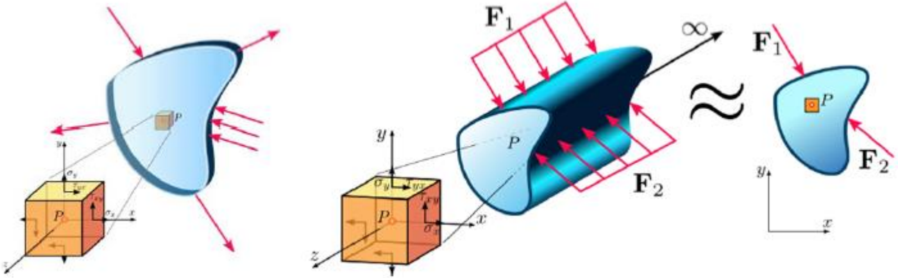
In the case of viscoelastic material the harmonic equation of motion fulfills the formula:

$$-\rho\omega^2\mathbf{u} - \nabla \cdot \mathbf{S} = \mathbf{F}e^{i\phi}. \quad (11)$$

$$\mathbf{S} = \mathbf{S}_{\mathrm{ad}} + \mathbf{C}:\boldsymbol{\varepsilon}_{\mathrm{el}} - (trace(\mathbf{C}:\boldsymbol{\varepsilon}_{\mathrm{el}})/3 + p_w)\mathbf{I} \quad (12)$$

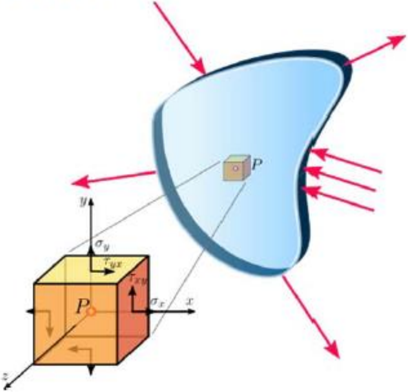
The trace of an n-by-n square matrix **A** is defined as the sum of the elements on the main diagonal (the diagonal from the upper left to the lower right): $trace(\mathbf{A}) = \sum_i A_{ii}$.

Plane Stress and Plane Strain

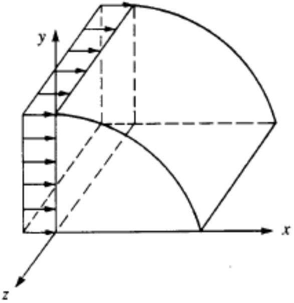


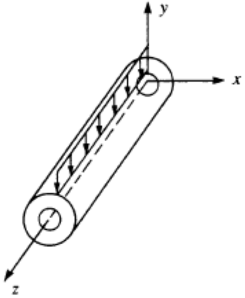
The two-dimensional element is extremely important for:

(1) ***Plane stress analysis***, which includes problems such as plates with holes, fillets, or other changes in geometry that are loaded in their plane resulting in local stress concentrations.



(2) ***Plane strain analysis***, which includes problems such as a long underground box culvert subjected to a uniform load acting constantly over its length or a long cylindrical control rod subjected to a load that remains constant over the rod length (or depth).





Plane Stress

Plane stress is defined to be ***a state of stress in which the normal stress and the shear stresses directed perpendicular to the plane are assumed to be zero***.

That is, the normal stress σ_z and the shear stresses τ_{xz} and τ_{yz} are assumed to be zero.

Generally, members that are thin (those with a small *z* dimension compared to the in-plane *x* and *y* dimensions) and whose loads act only in the *x*-*y* plane can be considered to be under plane stress.

Plane Strain

Plane strain is defined to be ***a state of strain in which the strain normal to the x-y plane ε_z and the shear strains γ_{xz} and γ_{yz} are assumed to be zero.***

The assumptions of plane strain are realistic for long bodies (say, in the *z* direction) with constant cross-sectional area subjected to loads that act only in the *x* and/or *y* directions and do not vary in the *z* direction.

Two-Dimensional State of Stress and Strain

For **plane stress,** the stresses σ_z, τ_{xz}, and τ_{yz} are assumed to be zero. The stress-strain relationship is:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 0.5(1-\nu) \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = [D] \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \qquad [D] = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 0.5(1-\nu) \end{bmatrix}$$

is called the ***stress-strain matrix*** (or the ***constitutive matrix***), E is the modulus of elasticity, and ν is Poisson's ratio.

Two-Dimensional State of Stress and Strain

For **plane strain**, the strains ε_z, γ_{xz}, and γ_{yz} are assumed to be zero. The stress-strain relationship is:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & 0.5-\nu \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = [D] \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \qquad [D] = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & 0.5-\nu \end{bmatrix}$$

is called the ***stress-strain matrix*** (or the ***constitutive matrix***), *E* is the modulus of elasticity, and ν is Poisson's ratio.

The limits of Poisson's ratio for isotropic solids possess fundamental significance. Shape is preserved at the lower limit of $v = -1$ (applicable for both 3D and 2D). Volume is preserved at the upper limit $v = 1/2$ (for 3D) while area is preserved at the upper limit of $v = 1$ (for 2D). It is now of interest, though not in a practical sense, to present the bounds of Poisson's ratio under 1D, 2D and 3D analyses as

$$\begin{array}{ll} v = 0; & d = 1 \\ -1 \leq v \leq 1; & d = 2 \\ -1 \leq v \leq 1/2; & d = 3 \end{array} \qquad (3.2.17)$$

whereby d = 1, 2, 3 refer to the number of dimensions. Of course the so-called "bound" for d = 1 is not a bound but this has been included for the sake of completeness. Alternatively, the bounds for 2D and 3D can be combined to give

$$\begin{array}{ll} v = 0; & d = 1 \\ -1 \leq v \leq \frac{1}{d-1}; & d = 2, 3 \end{array} \cdot \qquad (3.2.18)$$

In addition to the Poisson's ratio bounds based on 3D analysis, it is possible to obtain the Poisson's ratio bounds for 2D. The upper bound of Poisson's ratio for 2D case can be performed either on the basis of plane strain or plane stress. In addition to $\sigma_{ij} = -p$; $(i = j)$ and $\sigma_{ij} = 0$; $(i \neq j)$ for hydrostatic pressure, the plane strain condition requires that $e_{33} = 0$. Of course the plane strain condition also implies $e_{23} = e_{31} = 0$ but these have no effect on our calculation. From Hooke's Law in 2D,

$$e_{11} = e_{22} \propto \frac{p}{E}(v - 1). \tag{3.2.13}$$

Since $e_{11} = e_{22} \leq 0$ due to the hydrostatic pressure and $E \geq 0$, we have $v - 1 \leq 0$ or

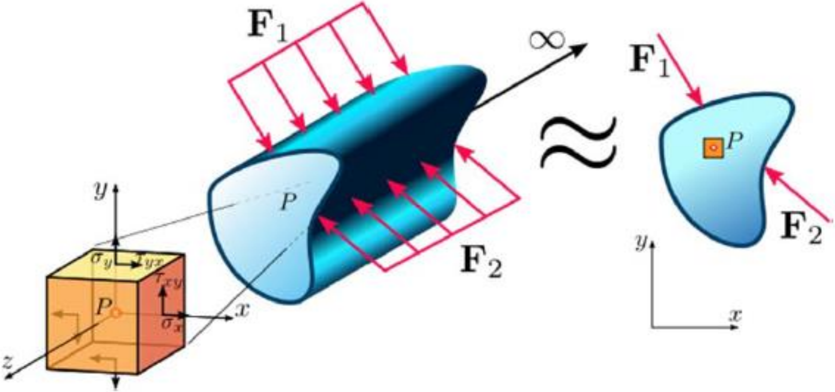
$$v \leq 1. \tag{3.2.14}$$

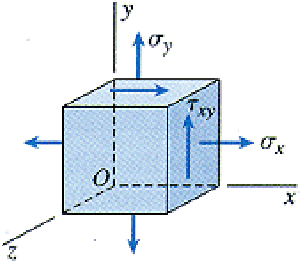
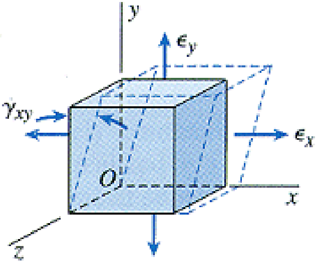
As before, the imposition of $e_{11} = e_{22} \leq 0$ arising from hydrostatic pressure and $E \geq 0$ leads to Eq. (3.2.14). Whether by plane strain ($e_{33} = 0$) or by plane stress ($\sigma_{33} = 0$), the strain energy for 2D analysis is common

$$U \propto \frac{p^2}{E}(1 - \nu) \tag{3.2.16}$$

because $\sigma_{33}e_{33} = 0$ for both cases under hydrostatic pressure. On the basis of $U \geq 0$ and $E \geq 0$, Eq. (3.2.14) is recovered for 2D analysis. Practically, the assumption of plane strain is more plausible since it is not possible to impose plane stress condition under hydrostatic pressure. The lower limit for the Poisson's ratio in 2D analysis is similar to that of 3D, because the condition of simple shear has only one stress component $\sigma_{23} = \tau$ regardless of 3D or 2D analyses.

Plane Strain Equations



	Plane stress	Plane strain
		
Stresses	$\sigma_z = 0$ $\tau_{xz} = 0$ $\tau_{yz} = 0$ σ_x, σ_y, and τ_{xy} may have nonzero values	$\tau_{xz} = 0$ $\tau_{yz} = 0$ σ_x, σ_y, σ_z, and τ_{xy} may have nonzero values
Strains	$\gamma_{xz} = 0$ $\gamma_{yz} = 0$ ϵ_x, ϵ_y, ϵ_z, and γ_{xy} may have nonzero values	$\epsilon_z = 0$ $\gamma_{xz} = 0$ $\gamma_{yz} = 0$ ϵ_x, ϵ_y, and γ_{xy} may have nonzero values