The Telegraph Equation

Introduction

This model examines how telegraph wire transmits a pulse of voltage using the *telegraph equation*. The telegraph equation models mixtures between diffusion and wave propagation by introducing a term that accounts for effects of finite velocity to a standard heat or mass transport equation.

This example models a small section of a telegraph wire and contains a study of the pulse of voltage moving along it. A parametric analysis provides results showing the shape of the pulse with varying damping coefficients.

Model Definition

The model is simple to define. The geometry is a one-dimensional line of length 1. To model the pulse, the initial condition is a bell-shaped voltage distribution. The boundary conditions define the flux at both ends of the wire section, which allows the voltage to vary freely.

DOMAIN EQUATIONS

The telegraph equation is the following:

$$u_{tt} + (\alpha + \beta)u_t + \alpha\beta u = c^2 u_{xx}$$

where:

- α and β are positive constants.
- c is the transport velocity.
- u is the voltage (the dependent variable).

The model begins with the values $\alpha = \beta = 0.25$ and $c = 1$.

BOUNDARY CONDITIONS

The boundary conditions at both ends are homogeneous Neumann conditions:

$$u_x(t, 0) = 0$$

$$u_x(t, 1) = 0$$

INITIAL CONDITION

The following equations for the initial condition describe a bell-shaped pulse with the highest point at 0.2 and a base width of 0.4:

$$u(0, x) = e^{-3\left(\frac{x}{0.2} - 1\right)^2}$$

$$u_t(0, x) = 0$$

Results

The figure below shows the results of the first simulation. It is clear that the pulse gets smoother as it propagates along the wire section. Figure 5-10 shows the shape of the pulse at t=0, 0.5, and 1:

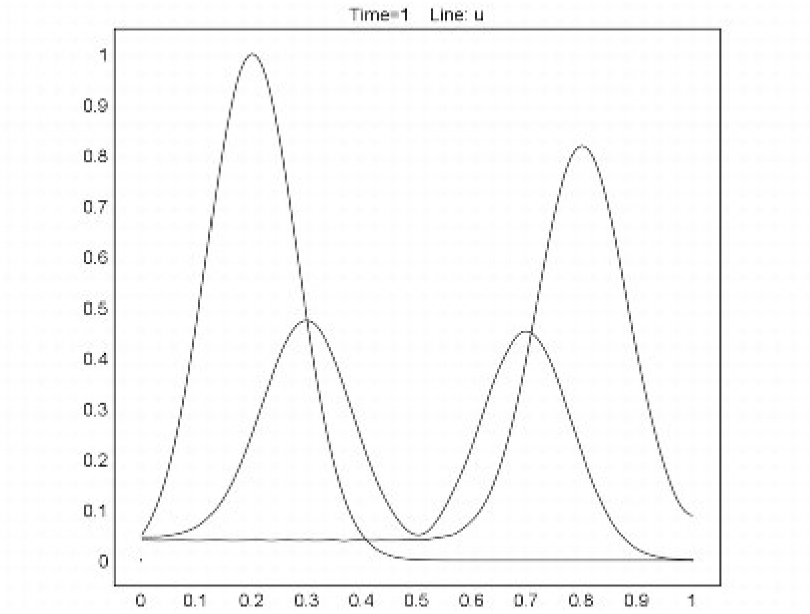


Figure 5-10: Shape of the pulse at t = 0, 0.5, and 1: $\alpha+\beta$ *= 0.5.*

Small values of the term αβ result in a smoother pulse compared to larger values, while the term α+β sets the amount of damping. The following plots shows the influence of the term α+β on the damping. A value of α+β = 1 yields the pulse in Figure 5-11 at *t* = 0, 0.5, and 1:

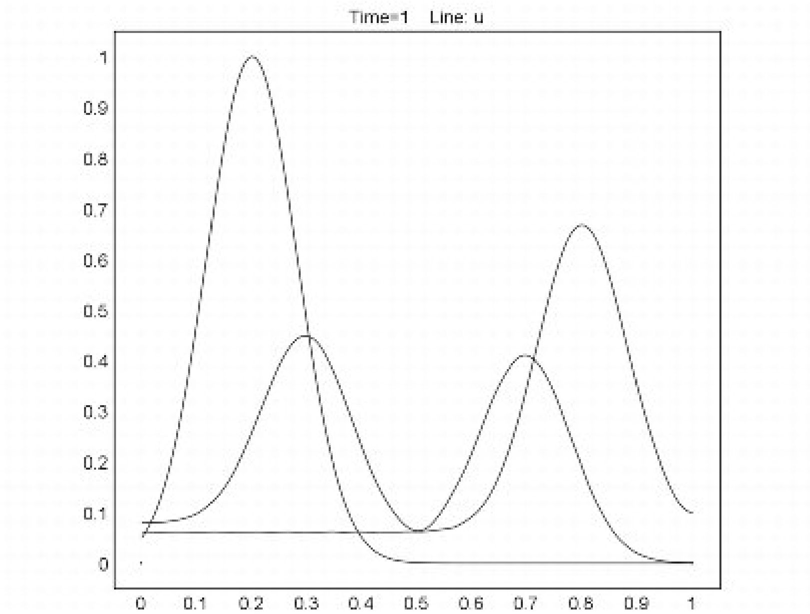


Figure 5-11: Shape of the pulse at t = 0, 0.5, and 1: $\alpha+\beta$ *= 1.*

In the figure above the height of the pulse decreases only slightly from the initial value. In Figure 5-12, the decrease in height is more pronounced owing to a damping term that is four times as large as the one used for Figure 5-11.

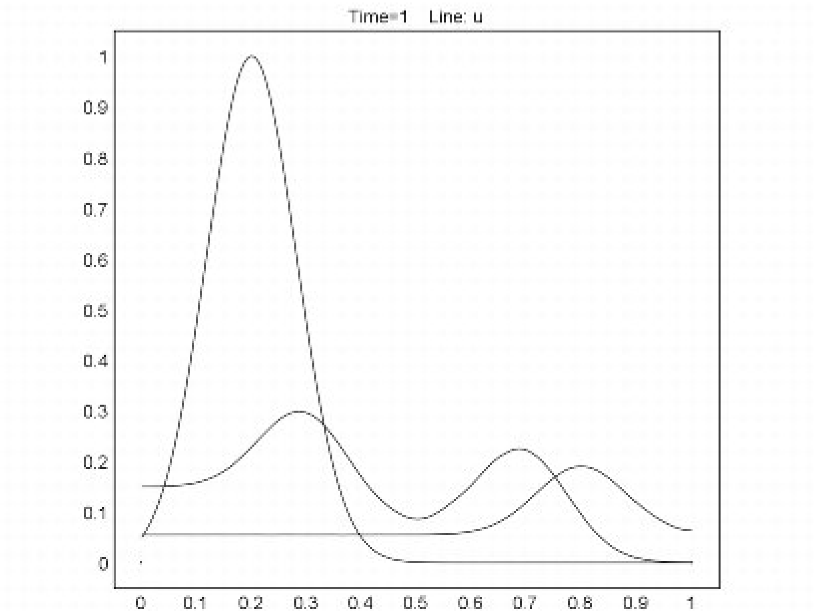


Figure 5-12: Shape of the pulse at t = 0, 0.5, and 1: $\alpha+\beta = 2$.

Applying the telegraph equation to 2D and 3D models follows the same protocol shown here but produces a more complex systems of equations.

Modeling in COMSOL Multiphysics

To set up the telegraph equation, use a Coefficient Form PDE application mode for time-dependent analysis.

Model Library Path: `COMSOL_Multiphysics/Equation-Based_Models/telegraph_equation`

Modeling Using the Graphical User Interface

MODEL NAVIGATOR

1 Select **1D** in the **Space dimension** list.

2 In the list of application modes, browse to **COMSOL Multiphysics>PDE Modes>PDE, Coefficient Form**.

3 Select **Time-dependent analysis, wave type**. Make sure that **Lagrange - Quadratic** elements are selected in the **Element** list.

4 Click **OK**.

OPTIONS AND SETTINGS

1 From the **Options** menu, choose **Constants**.

2 Enter the following constants in the **Constants** dialog box:

NAME	EXPRESSION
c	1
alpha	0.25
beta	0.25

GEOMETRY MODELING

1 Click the **Line** button and draw a line of length 1 from 0 to 1 on the x-axis.

2 Click the **Zoom Extents** toolbar button.

PHYSICS SETTINGS

Boundary Conditions

1 From the **Physics** menu, choose **Boundary Settings**.

2 Enter boundary coefficients as follows.

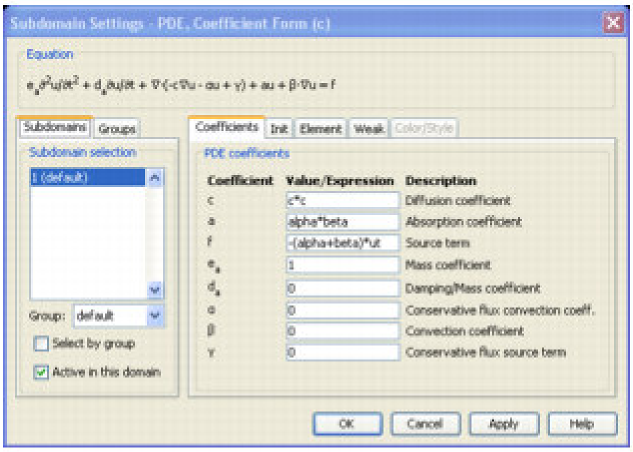
BOUNDARY	**1–2**
Type	Neumann

Subdomain Settings

1 From the **Physics** menu, choose **Subdomain Settings**.

2 On the **Coefficients** page in the **Subdomain Settings** dialog box, enter these PDE coefficients (the e_a and d_a coefficients are the default values):

SUBDOMAIN	**1**
c	`c*c`
a	`alpha*beta`
f	`-(alpha+beta)*ut`
d_a	`0`
e_a	`1`



3 Click the **Init** tab and enter the following initial conditions:

SUBDOMAIN	**1**
$u(t_0)$	`exp(-3*(x/0.2-1)^2)`
$u_t(t_0)$	`0`

MESH GENERATION

Initialize the mesh and refine it once.

COMPUTING THE SOLUTION

1 From the **Solve** menu, choose **Solver Parameters**.

2 Click the **Time Stepping** tab.

3 Select the **Manual tuning of step size** check box. Type `0.002` in the **Maximum time step** edit field. To get a good solution to the telegraph equation with this mesh, a time step of around 0.002 is sufficient. It means that the solver takes about 500 steps.

4 Click **OK**.

5 Click the **Solve** toolbar button to run the analysis.

POSTPROCESSING AND VISUALIZATION

1 From the **Postprocessing** menu, choose **Plot Parameters**.

2 Select the **Keep current plot** check box and plot the solution at time 0, 0.5, and 1.

3 From the **Options** menu, choose **Constants**.

4 In the **Constants** dialog box, change `alpha` and `beta` to 0.5.

5 Click **OK**.

6 Click the **Solve** button.

Continue to investigate the effect of the α and β coefficients by changing their values.