Y.2. Finite element model of two-phase flow

The governing equations for two-phase flow summarized in Chapter must first be discrtetized in space.
The following shape functions will be used to approximate the field variables

$$v_1 = \sum_{i=1}^{nu} N_i v_{1i}\ ,\ v_2 = \sum_{i=1}^{nu} N_i v_{2i}\ ,\ v_3 = \sum_{i=1}^{nu} N_i v_{3i}\ ,\ p = \sum_{i=1}^{np} NP_i p_i\ ,\ C = \sum_{i=1}^{nc} NC_i C_i\,. \quad \text{(x.y)}$$

Substitution of (x.y) into the virtual work equation yields

$$\int_\Omega N_i \frac{\partial N_j}{\partial x_1}\left\{(v_1)_{n+1}^r\right\}d\Omega + \int_\Omega N_i \frac{\partial N_j}{\partial x_2}\left\{(v_2)_{n+1}^r\right\}d\Omega + \int_\Omega N_i \frac{\partial N_j}{\partial x_3}\left\{(v_3)_{n+1}^r\right\}d\Omega = 0 \quad \text{(x.y)}$$

$$\begin{aligned}
&\int_\Omega N_i(\rho)_{n+1}^{r-1}\left(\frac{N_j}{\Delta t} + (v_1)_{n+1}^{r-1}\frac{\partial N_j}{\partial x_1} + (v_2)_{n+1}^{r-1}\frac{\partial N_j}{\partial x_2} + (v_3)_{n+1}^{r-1}\frac{\partial N_j}{\partial x_3}\right)\left\{(v_1)_{n+1}^r\right\}d\Omega \\
&+\int_\Omega N_i \frac{\partial NP_j}{\partial x_1}\left\{(p)_{n+1}^r\right\}d\Omega - \int_\Omega N_i(\eta)_{n+1}^{r-1}\left(\frac{\partial^2 N_j}{\partial x_1^2} + \frac{\partial^2 N_j}{\partial x_2^2} + \frac{\partial^2 N_j}{\partial x_3^2}\right)\left\{(v_1)_{n+1}^r\right\}d\Omega = \\
&= \int_\Omega N_i \left(\begin{aligned}&(f_1)_{n+1}^{r-1} + (\rho)_{n+1}^{r-1}\frac{(v_1)_n}{\Delta t} + 2\frac{\partial(\eta)_{n+1}^{r-1}}{\partial x_1}\frac{\partial(v_1)_{n+1}^{r-1}}{\partial x_1} + \\ &+\frac{\partial(\eta)_{n+1}^{r-1}}{\partial x_2}\left(\frac{\partial(v_1)_{n+1}^{r-1}}{\partial x_2} + \frac{\partial(v_2)_{n+1}^{r-1}}{\partial x_1}\right) + \frac{\partial(\eta)_{n+1}^{r-1}}{\partial x_3}\left(\frac{\partial(v_1)_{n+1}^{r-1}}{\partial x_3} + \frac{\partial(v_3)_{n+1}^{r-1}}{\partial x_1}\right)\end{aligned}\right)d\Omega
\end{aligned} \quad \text{(x.y)}$$

$$\begin{aligned}
&\int_\Omega N_i(\rho)_{n+1}^{r-1}\left(\frac{N_j}{\Delta t} + (v_1)_{n+1}^{r-1}\frac{\partial N_j}{\partial x_1} + (v_2)_{n+1}^{r-1}\frac{\partial N_j}{\partial x_2} + (v_3)_{n+1}^{r-1}\frac{\partial N_j}{\partial x_3}\right)\left\{(v_2)_{n+1}^r\right\}d\Omega \\
&+\int_\Omega N_i \frac{\partial NP_j}{\partial x_2}\left\{(p)_{n+1}^r\right\}d\Omega - \int_\Omega N_i(\eta)_{n+1}^{r-1}\left(\frac{\partial^2 N_j}{\partial x_1^2} + \frac{\partial^2 N_j}{\partial x_2^2} + \frac{\partial^2 N_j}{\partial x_3^2}\right)\left\{(v_2)_{n+1}^r\right\}d\Omega =
\end{aligned} \quad \text{(x.y)}$$

$$
= \int_{\Omega} N_i \begin{pmatrix} (f_2)_{n+1}^{r-1} + (\rho)_{n+1}^{r-1} \dfrac{(v_2)_n}{\Delta t} + \dfrac{\partial(\eta)_{n+1}^{r-1}}{\partial x_1}\left(\dfrac{\partial (v_1)_{n+1}^{r-1}}{\partial x_2} + \dfrac{\partial (v_2)_{n+1}^{r-1}}{\partial x_1} \right) + \\ + 2\dfrac{\partial(\eta)_{n+1}^{r-1}}{\partial x_2}\dfrac{\partial (v_2)_{n+1}^{r-1}}{\partial x_2} + \dfrac{\partial(\eta)_{n+1}^{r-1}}{\partial x_3}\left(\dfrac{\partial (v_2)_{n+1}^{r-1}}{\partial x_3} + \dfrac{\partial (v_3)_{n+1}^{r-1}}{\partial x_2} \right) \end{pmatrix} d\Omega
$$

$$
\begin{aligned}
&\int_{\Omega} N_i (\rho)_{n+1}^{r-1} \left(\frac{N_j}{\Delta t} + (v_1)_{n+1}^{r-1} \frac{\partial N_j}{\partial x_1} + (v_2)_{n+1}^{r-1} \frac{\partial N_j}{\partial x_2} + (v_3)_{n+1}^{r-1} \frac{\partial N_j}{\partial x_3} \right) \left\{ (v_3)_{n+1}^{r} \right\} d\Omega \\
&+ \int_{\Omega} N_i \frac{\partial NP_j}{\partial x_3} \left\{ (p)_{n+1}^{r} \right\} d\Omega - \int_{\Omega} N_i (\eta)_{n+1}^{r-1} \left(\frac{\partial^2 N_j}{\partial x_1^2} + \frac{\partial^2 N_j}{\partial x_2^2} + \frac{\partial^2 N_j}{\partial x_3^2} \right) \left\{ (v_3)_{n+1}^{r} \right\} d\Omega = \\
&= \int_{\Omega} N_i \begin{pmatrix} (f_3)_{n+1}^{r-1} + (\rho)_{n+1}^{r-1} \dfrac{(v_3)_n}{\Delta t} + \dfrac{\partial(\eta)_{n+1}^{r-1}}{\partial x_1}\left(\dfrac{\partial (v_3)_{n+1}^{r-1}}{\partial x_1} + \dfrac{\partial (v_1)_{n+1}^{r-1}}{\partial x_3} \right) + \\ + \dfrac{\partial(\eta)_{n+1}^{r-1}}{\partial x_2}\left(\dfrac{\partial (v_3)_{n+1}^{r-1}}{\partial x_2} + \dfrac{\partial (v_2)_{n+1}^{r-1}}{\partial x_3} \right) + 2\dfrac{\partial(\eta)_{n+1}^{r-1}}{\partial x_3}\dfrac{\partial (v_3)_{n+1}^{r-1}}{\partial x_3} \end{pmatrix} d\Omega
\end{aligned} \qquad \text{(x.y)}
$$

$$
\begin{aligned}
&\int_{\Omega} N_i \left(\frac{NC_j}{\Delta t} + (v_1)_{n+1}^{r} \frac{\partial NC_j}{\partial x_1} + (v_2)_{n+1}^{r} \frac{\partial NC_j}{\partial x_2} + (v_3)_{n+1}^{r} \frac{\partial NC_j}{\partial x_3} \right) \left\{ (C)_{n+1}^{r} \right\} d\Omega = \\
&= \int_{\Omega} N_i \frac{C_n}{\Delta t} d\Omega
\end{aligned} \qquad \text{(x.y)}
$$

By applying integration by parts to the integral expression of equations, we can obtain following expressions containing lower-order derivatives

$$
\begin{aligned}
&\int_{\Omega} N_i (\rho)_{n+1}^{r-1} \left(\frac{N_j}{\Delta t} + (v_1)_{n+1}^{r-1} \frac{\partial N_j}{\partial x_1} + (v_2)_{n+1}^{r-1} \frac{\partial N_j}{\partial x_2} + (v_3)_{n+1}^{r-1} \frac{\partial N_j}{\partial x_3} \right) \left\{ (v_1)_{n+1}^{r} \right\} d\Omega \\
&+ \int_{\Omega} N_i \frac{\partial NP_j}{\partial x_1} \left\{ (p)_{n+1}^{r} \right\} d\Omega - \int_{\Omega} (\eta)_{n+1}^{r-1} \left(\frac{\partial N_i}{\partial x_1}\frac{\partial N_j}{\partial x_1} + \frac{\partial N_i}{\partial x_2}\frac{\partial N_j}{\partial x_2} + \frac{\partial N_i}{\partial x_3}\frac{\partial N_j}{\partial x_3} \right) \left\{ (v_1)_{n+1}^{r} \right\} d\Omega =
\end{aligned} \qquad \text{(x.y)}
$$

$$= \int_{\Omega} N_i \left(\begin{array}{l} (f_1)_{n+1}^{r-1} + (\rho)_{n+1}^{r-1} \dfrac{(v_1)_n}{\Delta t} + 2 \dfrac{\partial (\eta)_{n+1}^{r-1}}{\partial x_1} \dfrac{\partial (v_1)_{n+1}^{r-1}}{\partial x_1} + \\ + \dfrac{\partial (\eta)_{n+1}^{r-1}}{\partial x_2} \left(\dfrac{\partial (v_1)_{n+1}^{r-1}}{\partial x_2} + \dfrac{\partial (v_2)_{n+1}^{r-1}}{\partial x_1} \right) + \dfrac{\partial (\eta)_{n+1}^{r-1}}{\partial x_3} \left(\dfrac{\partial (v_1)_{n+1}^{r-1}}{\partial x_3} + \dfrac{\partial (v_3)_{n+1}^{r-1}}{\partial x_1} \right) \end{array} \right) d\Omega$$

$$- \int_{\Gamma} N_i \frac{\partial (v_1)_{n+1}^{r}}{\partial \mathbf{n}} d\Gamma$$

$$\int_{\Omega} N_i (\rho)_{n+1}^{r-1} \left(\frac{N_j}{\Delta t} + (v_1)_{n+1}^{r-1} \frac{\partial N_j}{\partial x_1} + (v_2)_{n+1}^{r-1} \frac{\partial N_j}{\partial x_2} + (v_3)_{n+1}^{r-1} \frac{\partial N_j}{\partial x_3} \right) \left\{ (v_2)_{n+1}^{r} \right\} d\Omega$$

$$+ \int_{\Omega} N_i \frac{\partial NP_j}{\partial x_2} \left\{ (p)_{n+1}^{r} \right\} d\Omega - \int_{\Omega} (\eta)_{n+1}^{r-1} \left(\frac{\partial N_i}{\partial x_1} \frac{\partial N_j}{\partial x_1} + \frac{\partial N_i}{\partial x_2} \frac{\partial N_j}{\partial x_2} + \frac{\partial N_i}{\partial x_3} \frac{\partial N_j}{\partial x_3} \right) \left\{ (v_2)_{n+1}^{r} \right\} d\Omega =$$

$$= \int_{\Omega} N_i \left(\begin{array}{l} (f_2)_{n+1}^{r-1} + (\rho)_{n+1}^{r-1} \dfrac{(v_2)_n}{\Delta t} + \dfrac{\partial (\eta)_{n+1}^{r-1}}{\partial x_1} \left(\dfrac{\partial (v_1)_{n+1}^{r-1}}{\partial x_2} + \dfrac{\partial (v_2)_{n+1}^{r-1}}{\partial x_1} \right) + \\ + 2 \dfrac{\partial (\eta)_{n+1}^{r-1}}{\partial x_2} \dfrac{\partial (v_2)_{n+1}^{r-1}}{\partial x_2} + \dfrac{\partial (\eta)_{n+1}^{r-1}}{\partial x_3} \left(\dfrac{\partial (v_2)_{n+1}^{r-1}}{\partial x_3} + \dfrac{\partial (v_3)_{n+1}^{r-1}}{\partial x_2} \right) \end{array} \right) d\Omega \qquad \text{(x.y)}$$

$$- \int_{\Gamma} N_i \frac{\partial (v_2)_{n+1}^{r}}{\partial \mathbf{n}} d\Gamma$$

$$\int_{\Omega} N_i (\rho)_{n+1}^{r-1} \left(\frac{N_j}{\Delta t} + (v_1)_{n+1}^{r-1} \frac{\partial N_j}{\partial x_1} + (v_2)_{n+1}^{r-1} \frac{\partial N_j}{\partial x_2} + (v_3)_{n+1}^{r-1} \frac{\partial N_j}{\partial x_3} \right) \left\{ (v_3)_{n+1}^{r} \right\} d\Omega$$

$$+ \int_{\Omega} N_i \frac{\partial NP_j}{\partial x_3} \left\{ (p)_{n+1}^{r} \right\} d\Omega - \int_{\Omega} (\eta)_{n+1}^{r-1} \left(\frac{\partial N_i}{\partial x_1} \frac{\partial N_j}{\partial x_1} + \frac{\partial N_i}{\partial x_2} \frac{\partial N_j}{\partial x_2} + \frac{\partial N_i}{\partial x_3} \frac{\partial N_j}{\partial x_3} \right) \left\{ (v_3)_{n+1}^{r} \right\} d\Omega = \qquad \text{(x.y)}$$

$$= \int_{\Omega} N_i \left(\begin{aligned} &\left(f_3\right)_{n+1}^{r-1} + \left(\rho\right)_{n+1}^{r-1} \frac{\left(v_3\right)_n}{\Delta t} + \frac{\partial\left(\eta\right)_{n+1}^{r-1}}{\partial x_1} \left(\frac{\partial\left(v_3\right)_{n+1}^{r-1}}{\partial x_1} + \frac{\partial\left(v_1\right)_{n+1}^{r-1}}{\partial x_3} \right) + \\ &+ \frac{\partial\left(\eta\right)_{n+1}^{r-1}}{\partial x_2} \left(\frac{\partial\left(v_3\right)_{n+1}^{r-1}}{\partial x_2} + \frac{\partial\left(v_2\right)_{n+1}^{r-1}}{\partial x_3} \right) + 2 \frac{\partial\left(\eta\right)_{n+1}^{r-1}}{\partial x_3} \frac{\partial\left(v_3\right)_{n+1}^{r-1}}{\partial x_3} \end{aligned} \right) d\Omega$$

$$- \int_{\Gamma} N_i \frac{\partial\left(v_3\right)_{n+1}^{r}}{\partial \mathbf{n}} d\Gamma$$